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Uncertainty relation on Wigner–Yanase–Dyson skew information[☆]

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ABSTRACT

We give a trace inequality related to the uncertainty relation of Wigner–Yanase–Dyson skew information. This inequality corresponds to a generalization of the uncertainty relation derived by S. Luo (2005) [8] for the quantum uncertainty quantity excluding the classical mixture.

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1. Introduction

Wigner–Yanase skew information

$$\begin{aligned} I_{\rho}(H) &= \frac{1}{2} \operatorname{Tr}[(i[\rho^{1/2}, H])^2] \\ &= \operatorname{Tr}[\rho H^2] - \operatorname{Tr}[\rho^{1/2} H \rho^{1/2} H] \end{aligned}$$

was defined in [10]. This quantity can be considered as a kind of the degree for non-commutativity between a quantum state ρ and an observable H . Here we denote the commutator by $[X, Y] = XY - YX$. This quantity was generalized by Dyson

$$\begin{aligned} I_{\rho, \alpha}(H) &= \frac{1}{2} \operatorname{Tr}[(i[\rho^{\alpha}, H])(i[\rho^{1-\alpha}, H])] \\ &= \operatorname{Tr}[\rho H^2] - \operatorname{Tr}[\rho^{\alpha} H \rho^{1-\alpha} H], \quad \alpha \in [0, 1] \end{aligned}$$

which is known as the Wigner–Yanase–Dyson skew information. It is famous that the convexity of $I_{\rho, \alpha}(H)$ with respect to ρ was successfully proven by E.H. Lieb in [7]. From the physical point of view, an observable H is generally considered to be an unbounded operator, however in the present paper, unless otherwise stated, we consider $H \in B(\mathcal{H})$ represents the set of all bounded linear operators on the Hilbert space \mathcal{H} , as a mathematical interest. We also denote the set of all self-adjoint operators (observables) by $\mathcal{L}_h(\mathcal{H})$ and the set of all density operators (quantum states) by $\mathcal{S}(\mathcal{H})$ on the Hilbert space \mathcal{H} . The relation between the Wigner–Yanase skew information and the uncertainty relation was studied in [9]. Moreover the relation between the Wigner–Yanase–Dyson skew information and the uncertainty relation was studied in [5,11]. In our paper [11], we defined a generalized skew information and then derived a kind of an uncertainty relation. In Section 2, we discuss various properties of the Wigner–Yanase–Dyson skew information. Finally in Section 3, we give our main result and its proof.

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2. Trace inequalities of Wigner–Yanase–Dyson skew information

We review the relation between the Wigner–Yanase skew information and the uncertainty relation. In quantum mechanical system, the expectation value of an observable H in a quantum state ρ is expressed by $\text{Tr}[\rho H]$. It is natural that the variance for a quantum state ρ and an observable H is defined by $V_\rho(H) = \text{Tr}[\rho(H - \text{Tr}[\rho H]I)^2] = \text{Tr}[\rho H^2] - \text{Tr}[\rho H]^2$. It is famous that we have

$$V_\rho(A)V_\rho(B) \geq \frac{1}{4} |\text{Tr}[\rho[A, B]]|^2 \quad (2.1)$$

for a quantum state ρ and two observables A and B (see [4]). The further strong results was given by Robertson and Schrödinger

$$V_\rho(A)V_\rho(B) - |\text{Cov}_\rho(A, B)|^2 \geq \frac{1}{4} |\text{Tr}[\rho[A, B]]|^2,$$

where the covariance is defined by $\text{Cov}_\rho(A, B) = \text{Tr}[\rho(A - \text{Tr}[\rho A]I)(B - \text{Tr}[\rho B]I)]$. However, the uncertainty relation for the Wigner–Yanase skew information failed. (See [5,9,11].)

$$I_\rho(A)I_\rho(B) \geq \frac{1}{4} |\text{Tr}[\rho[A, B]]|^2.$$

Recently, S. Luo introduced the quantity $U_\rho(H)$ representing a quantum uncertainty excluding the classical mixture:

$$U_\rho(H) = \sqrt{V_\rho(H)^2 - (V_\rho(H) - I_\rho(H))^2}, \quad (2.2)$$

then he derived the uncertainty relation on $U_\rho(H)$ in [8]:

$$U_\rho(A)U_\rho(B) \geq \frac{1}{4} |\text{Tr}[\rho[A, B]]|^2. \quad (2.3)$$

Note that we have the following relation

$$0 \leq I_\rho(H) \leq U_\rho(H) \leq V_\rho(H). \quad (2.4)$$

The inequality (2.3) is a refinement of the inequality (2.1) in the sense of (2.4). In this section, we study one-parameter extended inequality for the inequality (2.3).

Definition 2.1. For $0 \leq \alpha \leq 1$, a quantum state ρ and an observable H , we define the Wigner–Yanase–Dyson skew information

$$\begin{aligned} I_{\rho, \alpha}(H) &= \frac{1}{2} \text{Tr}[(i[\rho^\alpha, H_0])(i[\rho^{1-\alpha}, H_0])] \\ &= \text{Tr}[\rho H_0^2] - \text{Tr}[\rho^\alpha H_0 \rho^{1-\alpha} H_0] \end{aligned} \quad (2.5)$$

and we also define

$$\begin{aligned} J_{\rho, \alpha}(H) &= \frac{1}{2} \text{Tr}[\{\rho^\alpha, H_0\}\{\rho^{1-\alpha}, H_0\}] \\ &= \text{Tr}[\rho H_0^2] + \text{Tr}[\rho^\alpha H_0 \rho^{1-\alpha} H_0], \end{aligned} \quad (2.6)$$

where $H_0 = H - \text{Tr}[\rho H]I$ and we denote the anti-commutator by $\{X, Y\} = XY + YX$.

Note that we have

$$\frac{1}{2} \text{Tr}[(i[\rho^\alpha, H_0])(i[\rho^{1-\alpha}, H_0])] = \frac{1}{2} \text{Tr}[(i[\rho^\alpha, H])(i[\rho^{1-\alpha}, H])]$$

but we have

$$\frac{1}{2} \text{Tr}[\{\rho^\alpha, H_0\}\{\rho^{1-\alpha}, H_0\}] \neq \frac{1}{2} \text{Tr}[\{\rho^\alpha, H\}\{\rho^{1-\alpha}, H\}].$$

Then we have the following inequalities:

$$I_{\rho, \alpha}(H) \leq I_\rho(H) \leq J_\rho(H) \leq J_{\rho, \alpha}(H), \quad (2.7)$$

since we have $\text{Tr}[\rho^{1/2} H \rho^{1/2} H] \leq \text{Tr}[\rho^\alpha H \rho^{1-\alpha} H]$. (See [1,2] for example.) If we define

$$U_{\rho, \alpha}(H) = \sqrt{V_\rho(H)^2 - (V_\rho(H) - I_{\rho, \alpha}(H))^2}, \quad (2.8)$$

as a direct generalization of Eq. (2.2), then we have

$$0 \leq I_{\rho,\alpha}(H) \leq U_{\rho,\alpha}(H) \leq U_{\rho}(H) \quad (2.9)$$

due to the first inequality of (2.7). We also have

$$U_{\rho,\alpha}(H) = \sqrt{I_{\rho,\alpha}(H) J_{\rho,\alpha}(H)}.$$

From the inequalities (2.4), (2.8), (2.9), our situation is that we have

$$0 \leq I_{\rho,\alpha}(H) \leq I_{\rho}(H) \leq U_{\rho}(H)$$

and

$$0 \leq I_{\rho,\alpha}(H) \leq U_{\rho,\alpha}(H) \leq U_{\rho}(H).$$

Our concern is to show an uncertainty relation with respect to $U_{\rho,\alpha}(H)$ as a direct generalization of the inequality (2.3) such that

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) \geq \frac{1}{4} |\text{Tr}[\rho[A, B]]|^2. \quad (2.10)$$

On the other hand, we introduced a generalized Wigner–Yanase skew information which is a generalization of the inequality (2.10), but different from the Wigner–Yanase–Dyson skew information defined in (2.5) and gave the following theorem in [3].

Theorem 2.1. For $0 \leq \alpha \leq 1$, a quantum state ρ and an observable H , we define a generalized Wigner–Yanase skew information by

$$K_{\rho,\alpha}(H) = \frac{1}{2} \text{Tr} \left[\left(i \left[\frac{\rho^{\alpha} + \rho^{1-\alpha}}{2}, H_0 \right] \right)^2 \right]$$

and we also define

$$L_{\rho,\alpha}(H) = \frac{1}{2} \text{Tr} \left[\left(i \left\{ \frac{\rho^{\alpha} + \rho^{1-\alpha}}{2}, H_0 \right\} \right)^2 \right],$$

and

$$W_{\rho,\alpha}(H) = \sqrt{K_{\rho,\alpha}(H) L_{\rho,\alpha}(H)}.$$

Then for a quantum state ρ and observables A, B and $\alpha \in [0, 1]$, we have

$$W_{\rho,\alpha}(A)W_{\rho,\alpha}(B) \geq \frac{1}{4} \left| \text{Tr} \left[\left(\frac{\rho^{\alpha} + \rho^{1-\alpha}}{2} \right)^2 [A, B] \right] \right|^2.$$

3. Main theorem

We give the main theorem as follows;

Theorem 3.1. For a quantum state ρ and observables A, B and $0 \leq \alpha \leq 1$, we have

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) \geq \alpha(1-\alpha) |\text{Tr}[\rho[A, B]]|^2. \quad (3.1)$$

We use the several lemmas to prove Theorem 3.1. By spectral decomposition, there exists an orthonormal basis $\{\phi_1, \phi_2, \dots\}$ consisting of eigenvectors of ρ . Let $\lambda_1, \lambda_2, \dots$ be the corresponding eigenvalues, where $\sum_{i=1}^{\infty} \lambda_i = 1$ and $\lambda_i \geq 0$. Thus, ρ has a spectral representation

$$\rho = \sum_{i=1}^{\infty} \lambda_i |\phi_i\rangle \langle \phi_i|. \quad (3.2)$$

Lemma 3.1.

$$I_{\rho,\alpha}(H) = \sum_{i < j} (\lambda_i + \lambda_j - \lambda_i^{\alpha} \lambda_j^{1-\alpha} - \lambda_i^{1-\alpha} \lambda_j^{\alpha}) |\langle \phi_i | H_0 | \phi_j \rangle|^2.$$

Proof. By (3.2),

$$\rho H_0^2 = \sum_{i=1}^{\infty} \lambda_i |\phi_i\rangle \langle \phi_i| H_0^2.$$

Then

$$\text{Tr}[\rho H_0^2] = \sum_{i=1}^{\infty} \lambda_i \langle \phi_i | H_0^2 | \phi_i \rangle = \sum_{i=1}^{\infty} \lambda_i \|H_0 |\phi_i\rangle\|^2. \quad (3.3)$$

Since

$$\rho^\alpha H_0 = \sum_{i=1}^{\infty} \lambda_i^\alpha |\phi_i\rangle \langle \phi_i| H_0$$

and

$$\rho^{1-\alpha} H_0 = \sum_{i=1}^{\infty} \lambda_i^{1-\alpha} |\phi_i\rangle \langle \phi_i| H_0,$$

we have

$$\rho^\alpha H_0 \rho^{1-\alpha} H_0 = \sum_{i,j=1}^{\infty} \lambda_i^\alpha \lambda_j^{1-\alpha} |\phi_i\rangle \langle \phi_i| H_0 |\phi_j\rangle \langle \phi_j| H_0.$$

Thus

$$\begin{aligned} \text{Tr}[\rho^\alpha H_0 \rho^{1-\alpha} H_0] &= \sum_{i,j=1}^{\infty} \lambda_i^\alpha \lambda_j^{1-\alpha} \langle \phi_i | H_0 |\phi_j\rangle \langle \phi_j | H_0 |\phi_i\rangle \\ &= \sum_{i,j=1}^{\infty} \lambda_i^\alpha \lambda_j^{1-\alpha} |\langle \phi_i | H_0 |\phi_j\rangle|^2. \end{aligned} \quad (3.4)$$

From (2.5), (3.3), (3.4),

$$\begin{aligned} I_{\rho,\alpha}(H) &= \sum_{i=1}^{\infty} \lambda_i \|H_0 |\phi_i\rangle\|^2 - \sum_{i,j=1}^{\infty} \lambda_i^\alpha \lambda_j^{1-\alpha} |\langle \phi_i | H_0 |\phi_j\rangle|^2 \\ &= \sum_{i,j=1}^{\infty} (\lambda_i - \lambda_i^\alpha \lambda_j^{1-\alpha}) |\langle \phi_i | H_0 |\phi_j\rangle|^2 \\ &= \sum_{i < j} (\lambda_i + \lambda_j - \lambda_i^\alpha \lambda_j^{1-\alpha} - \lambda_i^{1-\alpha} \lambda_j^\alpha) |\langle \phi_i | H_0 |\phi_j\rangle|^2. \quad \square \end{aligned}$$

Lemma 3.2.

$$J_{\rho,\alpha}(H) \geq \sum_{i < j} (\lambda_i + \lambda_j + \lambda_i^\alpha \lambda_j^{1-\alpha} + \lambda_i^{1-\alpha} \lambda_j^\alpha) |\langle \phi_i | H_0 |\phi_j\rangle|^2.$$

Proof. By (2.6), (3.3), (3.4), we have

$$\begin{aligned} J_{\rho,\alpha}(H) &= \sum_{i=1}^{\infty} \lambda_i \|H_0 |\phi_i\rangle\|^2 + \sum_{i,j=1}^{\infty} \lambda_i^\alpha \lambda_j^{1-\alpha} |\langle \phi_i | H_0 |\phi_j\rangle|^2 \\ &= \sum_{i,j=1}^{\infty} (\lambda_i + \lambda_i^\alpha \lambda_j^{1-\alpha}) |\langle \phi_i | H_0 |\phi_j\rangle|^2 \\ &= 2 \sum_{i=1}^{\infty} \lambda_i |\langle \phi_i | H_0 |\phi_i\rangle|^2 + \sum_{i \neq j} (\lambda_i + \lambda_i^\alpha \lambda_j^{1-\alpha}) |\langle \phi_i | H_0 |\phi_j\rangle|^2 \end{aligned}$$

$$\begin{aligned}
&= 2 \sum_{i=1}^{\infty} \lambda_i |\langle \phi_i | H_0 | \phi_i \rangle|^2 + \sum_{i < j} (\lambda_i + \lambda_j + \lambda_i^\alpha \lambda_j^{1-\alpha} + \lambda_i^{1-\alpha} \lambda_j^\alpha) |\langle \phi_i | H_0 | \phi_j \rangle|^2 \\
&\geq \sum_{i < j} (\lambda_i + \lambda_j + \lambda_i^\alpha \lambda_j^{1-\alpha} + \lambda_i^{1-\alpha} \lambda_j^\alpha) |\langle \phi_i | H_0 | \phi_j \rangle|^2. \quad \square
\end{aligned}$$

Lemma 3.3. For any $t > 0$ and $0 \leq \alpha \leq 1$, the following inequality holds;

$$(1 - 2\alpha)^2(t - 1)^2 - (t^\alpha - t^{1-\alpha})^2 \geq 0. \quad (3.5)$$

Proof. If $\alpha = 0$ or $\frac{1}{2}$ or 1 , then it is clear that (3.5) is satisfied. Now we put

$$F(t) = (1 - 2\alpha)^2(t - 1)^2 - (t^\alpha - t^{1-\alpha})^2.$$

We have

$$F'(t) = 2(1 - 2\alpha)^2 t - 2\alpha t^{2\alpha-1} - 2(1 - \alpha)t^{1-2\alpha} + 8\alpha(1 - \alpha).$$

And we also have

$$F''(t) = 2(1 - 2\alpha)^2 - 2\alpha(2\alpha - 1)t^{2\alpha-2} - 2(1 - \alpha)(1 - 2\alpha)t^{-2\alpha}$$

and

$$\begin{aligned}
F'''(t) &= 4\alpha(1 - 2\alpha)(1 - \alpha)t^{-2\alpha-1} - 4\alpha(1 - 2\alpha)(1 - \alpha)t^{2\alpha-3} \\
&= 4\alpha(1 - 2\alpha)(1 - \alpha) \left(\frac{1}{t^{1+2\alpha}} - \frac{1}{t^{3-2\alpha}} \right).
\end{aligned}$$

If $\frac{1}{2} < \alpha < 1$, then $1 + 2\alpha > 3 - 2\alpha$. Then it is easy to show that $F'''(t) < 0$ for $t < 1$ and $F'''(t) > 0$ for $t > 1$. On the other hand if $0 < \alpha < \frac{1}{2}$, then $1 + 2\alpha < 3 - 2\alpha$. Then it is easy to show that $F'''(t) < 0$ for $t < 1$ and $F'''(t) > 0$ for $t > 1$. Since $F''(1) = 0$, we can get $F''(t) > 0$. Since $F'(1) = 0$, we also have $F'(t) < 0$ for $t < 1$ and $F'(t) > 0$ for $t > 1$. Since $F(1) = 0$, we finally get $F(t) \geq 0$ for all $t > 0$. Therefore we have (3.5). \square

Proof of Theorem 3.1. We put $t = \frac{\lambda_i}{\lambda_j}$ in (3.5). Then we have

$$(1 - 2\alpha)^2 \left(\frac{\lambda_i}{\lambda_j} - 1 \right)^2 - \left(\left(\frac{\lambda_i}{\lambda_j} \right)^\alpha - \left(\frac{\lambda_i}{\lambda_j} \right)^{1-\alpha} \right)^2 \geq 0.$$

And we get

$$(1 - 2\alpha)^2(\lambda_i - \lambda_j)^2 - (\lambda_i^\alpha \lambda_j^{1-\alpha} - \lambda_i^{1-\alpha} \lambda_j^\alpha)^2 \geq 0$$

and

$$(\lambda_i - \lambda_j)^2 - (\lambda_i^\alpha \lambda_j^{1-\alpha} - \lambda_i^{1-\alpha} \lambda_j^\alpha)^2 \geq 4\alpha(1 - \alpha)(\lambda_i - \lambda_j)^2$$

and

$$(\lambda_i + \lambda_j)^2 - (\lambda_i^\alpha \lambda_j^{1-\alpha} + \lambda_i^{1-\alpha} \lambda_j^\alpha)^2 \geq 4\alpha(1 - \alpha)(\lambda_i - \lambda_j)^2. \quad (3.6)$$

Since

$$\begin{aligned}
\text{Tr}[\rho[A, B]] &= \text{Tr}[\rho[A_0, B_0]] \\
&= 2i \text{Im} \text{Tr}[\rho A_0 B_0] \\
&= 2i \text{Im} \sum_{i < j} (\lambda_i - \lambda_j) \langle \phi_i | A_0 | \phi_j \rangle \langle \phi_j | B_0 | \phi_i \rangle \\
&= 2i \sum_{i < j} (\lambda_i - \lambda_j) \text{Im} \langle \phi_i | A_0 | \phi_j \rangle \langle \phi_j | B_0 | \phi_i \rangle, \\
|\text{Tr}[\rho[A, B]]| &= 2 \left| \sum_{i < j} (\lambda_i - \lambda_j) \text{Im} \langle \phi_i | A_0 | \phi_j \rangle \langle \phi_j | B_0 | \phi_i \rangle \right| \\
&\leq 2 \sum_{i < j} |\lambda_i - \lambda_j| |\text{Im} \langle \phi_i | A_0 | \phi_j \rangle \langle \phi_j | B_0 | \phi_i \rangle|.
\end{aligned}$$

Then we have

$$|Tr[\rho[A, B]]|^2 \leq 4 \left\{ \sum_{i < j} |\lambda_i - \lambda_j| |Im\langle \phi_i | A_0 | \phi_j \rangle \langle \phi_j | B_0 | \phi_i \rangle| \right\}^2.$$

By (3.6) and Schwarz inequality,

$$\begin{aligned} \alpha(1-\alpha)|Tr[\rho[A, B]]|^2 &\leq 4\alpha(1-\alpha) \left\{ \sum_{i < j} |\lambda_i - \lambda_j| |Im\langle \phi_i | A_0 | \phi_j \rangle \langle \phi_j | B_0 | \phi_i \rangle| \right\}^2 \\ &= \left\{ \sum_{i < j} 2\sqrt{\alpha(1-\alpha)} |\lambda_i - \lambda_j| |Im\langle \phi_i | A_0 | \phi_j \rangle \langle \phi_j | B_0 | \phi_i \rangle| \right\}^2 \\ &\leq \left\{ \sum_{i < j} 2\sqrt{\alpha(1-\alpha)} |\lambda_i - \lambda_j| |\langle \phi_i | A_0 | \phi_j \rangle| |\langle \phi_j | B_0 | \phi_i \rangle| \right\}^2 \\ &\leq \left\{ \sum_{i < j} \{(\lambda_i + \lambda_j)^2 - (\lambda_i^\alpha \lambda_j^{1-\alpha} + \lambda_i^{1-\alpha} \lambda_j^\alpha)^2\}^{1/2} |\langle \phi_i | A_0 | \phi_j \rangle| |\langle \phi_j | B_0 | \phi_i \rangle| \right\}^2 \\ &\leq \sum_{i < j} (\lambda_i + \lambda_j - \lambda_i^\alpha \lambda_j^{1-\alpha} - \lambda_i^{1-\alpha} \lambda_j^\alpha) |\langle \phi_i | A_0 | \phi_j \rangle|^2 \\ &\quad \times \sum_{i < j} (\lambda_i + \lambda_j + \lambda_i^\alpha \lambda_j^{1-\alpha} + \lambda_i^{1-\alpha} \lambda_j^\alpha) |\langle \phi_i | B_0 | \phi_j \rangle|^2. \end{aligned}$$

Then we have

$$I_{\rho, \alpha}(A) J_{\rho, \alpha}(B) \geq \alpha(1-\alpha) |Tr[\rho[A, B]]|^2.$$

We also have

$$I_{\rho, \alpha}(B) J_{\rho, \alpha}(A) \geq \alpha(1-\alpha) |Tr[\rho[A, B]]|^2.$$

Hence we have the final result (3.1). \square

Remark 3.1. We remark that (2.3) is derived by putting $\alpha = 1/2$ in (3.1). Then Theorem 3.1 is a generalization of the result of Luo [8].

Remark 3.2. We remark that Conjecture 2.3 in [3] does not hold in general. The conjecture is (2.10). A counterexample is given as follows. Let

$$\rho = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \alpha = \frac{1}{3}.$$

We have

$$I_{\rho, \alpha}(A) J_{\rho, \alpha}(B) = I_{\rho, \alpha}(B) I_{\rho, \alpha}(A) = 0.22457296 \dots$$

and $|Tr[\rho[A, B]]|^2 = 1$. These imply

$$U_{\rho, \alpha}(A) U_{\rho, \alpha}(B) = 0.22457296 \dots < \frac{1}{4} |Tr[\rho[A, B]]|^2 = 0.25.$$

On the other hand we have

$$U_{\rho, \alpha}(A) U_{\rho, \alpha}(B) > \alpha(1-\alpha) |Tr[\rho[A, B]]|^2 = 0.222222 \dots$$

We also give a counterexample for Conjecture 2.10 in [3]. The inequality

$$U_{\rho, \alpha}(A) U_{\rho, \alpha}(B) \geq \frac{1}{4} \left| Tr \left[\left(\frac{\rho^\alpha + \rho^{1-\alpha}}{2} \right)^2 [A, B] \right] \right|^2$$

is not correct in general, because $LHS = 0.22457296 \dots$, $RHS = 0.23828105995 \dots$.

Remark 3.3. In the recent literature another generalization for inequality (2.3) has been proved in [6] as follows; for any ρ , A , B and $0 \leq \alpha \leq 1$

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) \geq \frac{1}{4} |\text{Tr}[(\rho - \rho^{2\alpha-1})[A, B]]|^2.$$

However we gave the counter example for this inequality. Let

$$\rho = \begin{pmatrix} \frac{1}{64} & 0 & 0 \\ 0 & \frac{1}{16} & 0 \\ 0 & 0 & \frac{59}{64} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \alpha = \frac{3}{4}.$$

Then we have

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) = 0.00170898 \dots, \\ \frac{1}{4} |\text{Tr}[(\rho - \rho^{2\alpha-1})[A, B]]|^2 = 0.00610351 \dots$$

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